

ON HOFER ENERGY OF J -HOLOMORPHIC CURVES FOR ASYMPTOTICALLY CYLINDRICAL J

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Abstract

In this paper, we provide a bound for Hofer energy of punctured J -holomorphic curves in almost complex manifolds with asymptotically cylindrical ends. As an application, we prove that the number of times that any smooth J -holomorphic curve \tilde{u} passes through a fixed point in a closed symplectic manifold (M, ω') ¹ is bounded by a constant. The constant depends on the symplectic area of \tilde{u} , and does not depend on the domain Riemann surface and the map \tilde{u} . Here J is any ω' -compatible smooth almost complex structure on M . In particular, we do not require J to be integrable.

Key words. Asymptotically cylindrical, stable hamiltonian structure, J -holomorphic curve, Hofer energy, Holomorphic building.

1 Introduction

Hofer energy is introduced in [9] for J -holomorphic curves in symplectization of contact manifolds, and is generalized in [5] for J -holomorphic curves in the “almost complex manifolds with cylindrical ends”. Here “cylindrical” means that the almost complex structure J is invariant under translation. Hofer energy plays an essential role in the study of J -holomorphic curves in Symplectic Field Theory mainly because of the following two properties: (A) the asymptotic behavior of a J -holomorphic curve in a noncompact symplectic manifold can be controlled by requiring its Hofer energy to be finite, and hence a uniform Hofer energy bound gives a Symplectic Field Theory type of compactification of moduli spaces of J -holomorphic curves; on the other hand, (B) a uniform Hofer energy bound can be obtained by specifying the behavior the J -holomorphic curves at infinity and bounding their symplectic areas (see [9, 5]). In [2] the notion of Hofer energy and Property (A) are further generalized to include J -holomorphic curves in “almost complex manifolds with ASYMPTOTICALLY cylindrical ends”. Here “asymptotically cylindrical” means that the difference

¹Following the notation in [5] we save ω for something else.

between the almost complex structure J and a translation invariant one is exponentially small. In this paper, we prove Property (B) in this setting. Property (A) and property (B) together imply the expected useful compactness results in Symplectic Field Theory.

One of the main advantages of this generalization is that the asymptotically cylindrical J arises naturally. As an application, we prove that in a closed symplectic manifold (M, ω') with a fixed point $p \in M$ and a ω' -compatible almost complex structure J , the number of times that any J -holomorphic curve passes through p is uniformly bounded by a constant depending only on the symplectic area, M , ω' , J . This is closely related to a question asked in [6]. In [6], they study J -holomorphic curves with boundaries lying inside two clean intersecting Lagrangian submanifolds, and prove that the number of “boundary switches” at the intersecting loci is uniformly bounded by Hofer Energy. Their proof in an essential way relies on the additional requirement that the almost complex structure J is integrable near the intersecting loci. They ask to what extent their results are still true without assuming the integrability of J . In this paper, we provide a simple proof for the closed version of their result for arbitrary J . Namely, J -holomorphic curves we consider in this paper have no boundaries. In this case, “boundary switches” means that J -holomorphic curve passes p . Furthermore, the analysis developed in [2] and this paper can be carried out to include Lagrangians without difficulty.

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2 Asymptotically cylindrical almost complex structure

Let V_- be a smooth closed manifold of dimension $2N - 1$, J be a smooth almost complex structure on $W_- = \mathbb{R}^- \times V_-$, \mathbf{R} be the smooth vector field on W_- defined by $\mathbf{R} := J(\frac{\partial}{\partial r})$, and ξ be the subbundle of the tangent bundle TW_- defined by $\xi_{(r,v)} = (JT_v(\{r\} \times V_-)) \cap (T_v(\{r\} \times V_-))$, for $(r, v) \in W_-$. Then the tangent bundle TW_- splits as $TW_- = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$. Define the 1-forms λ and σ on W_- respectively by

$$\lambda(\xi) = 0 \quad \lambda(\frac{\partial}{\partial r}) = 0 \quad \lambda(\mathbf{R}) = 1, \quad (1)$$

$$\sigma(\xi) = 0 \quad \sigma(\frac{\partial}{\partial r}) = 1 \quad \sigma(\mathbf{R}) = 0. \quad (2)$$

Let $f_s : W_- \rightarrow W_-$ be the translation $f_s(r, v) = (r + s, v)$, for $s \leq 0$. We call a tensor on W_- translationally invariant if it is invariant under f_s .

Definition 1. Under the above notations, J is called asymptotically cylindrical at negative infinity, if there exists a 2-form ω on W such that the pair (J, ω) satisfies (AC1)-(AC7):

- (AC1) $i(\frac{\partial}{\partial r})\omega = 0 = i(\mathbf{R})\omega$.

- (AC2) $\omega|_{\xi}(\cdot, J\cdot)$ is a metric on ξ .
- (AC3) There exist a smooth translationally invariant almost complex structure $J_{-\infty}$ on W and constants $C_l, \delta_l \geq 0$, such that

$$\left\| (J - J_{-\infty})|_{(-\infty, r] \times V_-} \right\|_{C^l} \leq C_l e^{\delta_l r} \quad (3)$$

for all $r \leq 0$ and $l \in \mathbb{Z}_{\geq 0}$, where $\|\varphi\|_{C^l} := \sup_w \sum_{k=0}^l |\nabla^k \varphi(w)|$ and $|\cdot|$ is computed using a translationally invariant metric g_{W_-} on W_- , for example $g_{W_-} = dr^2 + g_{V_-}$, and ∇ is the corresponding Levi-Civita connection.

- (AC4) There exists a translationally invariant closed 2-form $\omega_{-\infty}$ on W_- and constants $C_l, \delta_l \geq 0$, such that

$$\left\| (\omega - \omega_{-\infty})|_{(-\infty, r] \times V_-} \right\|_{C^l} \leq C_l e^{\delta_l r} \quad (4)$$

for all $r \leq 0$ and $l \in \mathbb{Z}_{\geq 0}$.

- (AC5) The pair $(J_{-\infty}, \omega_{-\infty})$ satisfies (AC1) and (AC2).
- (AC6) $i(\mathbf{R}_{-\infty})d\lambda_{-\infty} = 0$, where $\mathbf{R}_{-\infty} := \lim_{s \rightarrow -\infty} f_s^* \mathbf{R}$, $\lambda_{-\infty} := \lim_{s \rightarrow -\infty} f_s^* \lambda$, and both limits exist by (AC3).
- (AC7) $\mathbf{R}_{-\infty}(r, v) = J_{-\infty}(\frac{\partial}{\partial r}) \in T_v(\{r\} \times V_-)$.

When we say J is asymptotically cylindrical, we choose ω without mentioning.

Similarly, we could define the notion of J being asymptotically cylindrical at positive infinity for $W_+ = \mathbb{R}^+ \times V_+$.

Remark 2. $(V_-, \omega_{-\infty})$ is a stable hamiltonian structure and $(\lambda_{-\infty}, J_{-\infty})$ is a framing of the stable hamiltonian structure (See [7] for the definition of stable hamiltonian structure. However, in this paper we do not need it).

Definition 3. We say an asymptotically cylindrical almost complex structure J is of contact type if $\omega_{-\infty} = d\lambda_{-\infty}$.

The following definition is the case considered in [9, 10, 4, 5].

Definition 4. We say J is a cylindrical almost complex structure, if J is an asymptotically cylindrical almost complex structure and both J and ω are translationally invariant.

By (AC6) and (AC7) we can see that $\mathbf{R}_{-\infty}$ is a translationally invariant vector field on W_- and it is tangent to each level set $\{r\} \times V_-$, so we can view $\mathbf{R}_{-\infty}$ as a vector field on V_- . Let ϕ^t be the flow of $\mathbf{R}_{-\infty}$ on V_- , i.e. $\phi^t : V_- \rightarrow V_-$ satisfies $\frac{d}{dt}\phi^t = \mathbf{R}_{-\infty} \circ \phi^t$. Then we have

$$\frac{d}{dt}[(\phi^t)^* \lambda_{-\infty}] = (\phi^t)^*(i_{\mathbf{R}_{-\infty}} d\lambda_{-\infty} + di_{\mathbf{R}_{-\infty}} \lambda_{-\infty}) = 0.$$

Thus ϕ^t preserves $\lambda_{-\infty}$ and hence $\xi_{-\infty}$. Similarly ϕ^t preserves $\omega_{-\infty}$.

Let's denote by \mathcal{P}_- the set of periodic trajectories, counting their multiples, of the vector field $\mathbf{R}_{-\infty}$ restricting to V_- . Notice that any smooth family of periodic trajectories from \mathcal{P}_- have the same period by Stokes' Theorem.

Definition 5. We say that an asymptotically cylindrical J is Morse-Bott if, for every $T > 0$ the subset $N_T \subseteq V_-$ formed by the closed trajectories from \mathcal{P}_- of period T is a smooth closed submanifold of V_- , such that the rank of $\omega_{-\infty}|_{N_T}$ is locally constant and $T_p N_T = \ker(d\phi^T - Id)_p$.

In this paper, we assume that J is Morse-Bott. For the application in section 4, it is easy to check that this requirement is satisfied.

Let $\Sigma := \mathbb{R}^- \times S^1$ be the half cylinder with standard almost complex structure j , and $\tilde{u} = (a, u) : (\Sigma, j) \rightarrow (W_-, J)$ be a J -holomorphic curve, i.e. $T\tilde{u} \circ j = J(\tilde{u}) \circ T\tilde{u}$. The ω -energy and λ -energy of \tilde{u} are defined as follows respectively

$$E_\omega(\tilde{u}) = \int_\Sigma \tilde{u}^* \omega,$$

$$E_\lambda(\tilde{u}) = \sup_{\phi \in \mathcal{C}} \int_\Sigma \tilde{u}^*(\phi(r)\sigma \wedge \lambda),$$

where $\mathcal{C} = \{\phi \in C^\infty(\mathbb{R}^-, [0, 1]) \mid \int_{-\infty}^0 \phi(x) dx = 1\}$, and λ and σ are defined as in (1) and (2). The Hofer energy of \tilde{u} is defined by

$$E(\tilde{u}) = E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

Let's equip $\mathbb{R}^- \times S^1$ with coordinate (s, t) . Here we view S^1 as \mathbb{R}/\mathbb{Z} . It is easy to check that $\tilde{u}^* \omega$ and $\tilde{u}^*(\phi(r)\sigma \wedge \lambda)$ are non-negative multiples of the volume form $ds \wedge dt$ on $\mathbb{R}^- \times S^1$. Actually,

$$\tilde{u}^* \omega = \omega(\pi_\xi \tilde{u}_s, J(\tilde{u}) \pi_\xi \tilde{u}_s) ds \wedge dt, \quad (5)$$

where π_ξ is the projection from $TW_- = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi$ to ξ , and

$$\tilde{u}^*(\phi(r)\sigma \wedge \lambda) = \phi(a) [\sigma(\tilde{u}_s)^2 + \lambda(\tilde{u}_s)^2] ds \wedge dt. \quad (6)$$

The following theorem is one of the most important theorems in [9, 10, 5, 4] for the case when J is cylindrical, and it is proved in the asymptotically cylindrical setting in [2].

Theorem 6. *Suppose that J is an asymptotically cylindrical almost complex structure on $W_- = \mathbb{R}^- \times V_-$. Let $\tilde{u} = (a, u) : \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \rightarrow W_-$ be a J -holomorphic curve with finite Hofer energy. Suppose that the image of \tilde{u} is unbounded in W_- . Then there exists a periodic orbit γ of $\mathbf{R}_{-\infty}$ of period $|T|$ with $T \neq 0$, such that*

$$\lim_{s \rightarrow -\infty} u(s, t) = \gamma(Tt) \quad (7)$$

$$\lim_{s \rightarrow -\infty} \frac{a(s, t)}{s} = T \quad (8)$$

in $C^\infty(S^1)$.

On the other hand, we have

Lemma 7. *Suppose that J is an asymptotically cylindrical almost complex structure on $W_- = \mathbb{R}^- \times V_-$, and $\tilde{u} = (a, u) : \mathbb{R}^- \times \mathbb{R}/\mathbb{Z} \rightarrow W_-$ is a J -holomorphic curve. Suppose that there exists a periodic orbit γ of $\mathbf{R}_{-\infty}$ of period $|T|$ such that*

$$\begin{aligned} \lim_{s \rightarrow -\infty} a(s, t) &= -\infty, \\ \lim_{s \rightarrow -\infty} u(s, t) &= \gamma(Tt). \end{aligned}$$

Then

$$\lim_{s \rightarrow -\infty} \frac{a(s, t)}{s} = T,$$

and Hofer energy $E(\tilde{u}) < \infty$.

Proof. This follows immediately from the proof of Theorem 2 in [2]. Namely, from the assumption, we could derive that the convergence in (7) and (8) is exponentially fast. Then it follows by definition and direct calculation that $E(\tilde{u}) < \infty$. \square

Remark 8. Theorem 6 and Lemma 7 also hold for W_+ .

3 Almost complex manifolds with asymptotically cylindrical ends

Now we introduce the notion of almost complex manifolds with asymptotically cylindrical ends.

Let (E, J) be a $2N$ dimensional noncompact almost complex manifold, and W_\pm be an open subset containing the positive (negative) end of E . Assume that W_\pm is diffeomorphic to $\mathbb{R}^\pm \times V_\pm$, where V_\pm is a $2N - 1$ dimensional closed manifold. Assume that there exists a J -compatible symplectic form ω' on E , and that $J|_{W_\pm}$ is an asymptotically cylindrical almost complex structure at positive (negative) infinity, then we say (E, J) is an almost complex manifold with asymptotically cylindrical positive (negative) ends.

Let \tilde{u} be a J -holomorphic map from a possibly punctured Riemann surface (Σ, j) to (E, J) , and then we define for $a \geq 0$,

$$E_{\text{symp}, a}(\tilde{u}) = \int_{\tilde{u}^{-1}(E \setminus W_+^a \cup W_-^a)} \tilde{u}^* \omega',$$

where $W_+^a := (a, +\infty) \times V_+ \subset W_+$, and $W_-^a := (-\infty, -a) \times V_- \subset W_-$.

$$E_\omega(\tilde{u}) = \int_{\tilde{u}^{-1}(W_+)} \tilde{u}^* \omega + \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^* \omega,$$

$$E_\lambda(\tilde{u}) = \sup_{\phi \in \mathcal{C}_+} \int_{\tilde{u}^{-1}(W_+)} \tilde{u}^*(\phi(r)\sigma \wedge \lambda) + \sup_{\phi \in \mathcal{C}_-} \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^*(\phi(r)\sigma \wedge \lambda),$$

where

$$\mathcal{C}_+ = \left\{ \phi \in C^\infty(\mathbb{R}^+, [0, 1]) \mid \int \phi = 1 \right\},$$

$$\mathcal{C}_- = \left\{ \phi \in C^\infty(\mathbb{R}^-, [0, 1]) \mid \int \phi = 1 \right\},$$

and

$$E_a(\tilde{u}) = E_{\text{symp},a}(\tilde{u}) + E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

If $\lim_{a \rightarrow +\infty} E_{\text{symp},a}(\tilde{u})$ is finite, we define

$$E_{\text{symp}}(\tilde{u}) = \lim_{a \rightarrow +\infty} E_{\text{symp},a}(\tilde{u})$$

and

$$E(\tilde{u}) = E_{\text{symp}}(\tilde{u}) + E_\omega(\tilde{u}) + E_\lambda(\tilde{u}).$$

To compactify the moduli space of J -holomorphic curves, we need to include holomorphic buildings (see [5]). There is no difference between almost complex manifolds with cylindrical ends and almost complex manifolds with asymptotically cylindrical ends when it comes to the definition of holomorphic buildings and the topology of the moduli space of holomorphic buildings. We also have the expected compactness theorem for the latter case.

Theorem 9. ([5] for cylindrical case; [2]) *For any $a \geq 0$, the moduli space of stable holomorphic buildings with uniformly bounded Hofer energy E_a , whose domains have a fixed number of arithmetic genus and a fixed number of marked points, is compact.*

The following theorem shows that in the contact case Hofer energy $E_a(\tilde{u})$ can be uniformly bounded by the Symplectic area $E_{\text{symp},a}(\tilde{u})$ and the periods of the periodic orbits γ_+ 's of $\mathbf{R}_{+\infty}$ that \tilde{u} converges to at positive infinity (compare to 9.2 in [5]).

Theorem 10. *Suppose (E, J) is an almost complex manifold with asymptotically cylindrical ends of contact type. There exists $C > 0$ and $a > 0$ such that for any finitely punctured Riemann surface (Σ, j) and any non-constant J -holomorphic curve $\tilde{u} : \Sigma \rightarrow E$ which converges to periodic orbits γ_\pm 's of $\mathbf{R}_{\pm\infty}$ around the punctures of Σ , we have $E_a(\tilde{u}) \leq C \left(\sum \int \gamma_+^* \lambda_{+\infty} + E_{\text{symp},a}(\tilde{u}) \right)$, where the summation is taken over all the periodic orbits γ_+ 's of $\mathbf{R}_{+\infty}$ to which \tilde{u} converges.*

Proof. Let us deal with the negative end W_- first.

For any $R > 0$, we pick $-\mathfrak{r} \in [-2R, -R]$ such that $-\mathfrak{r}$ is a regular value of $r \circ \tilde{u}$, where $r : W_- \rightarrow (-\infty, 0)$ is the projection map. Denote $A := \tilde{u}^{-1}((-\infty, -\mathfrak{r}] \times V_-) \subseteq \Sigma$ and $B_1 := \tilde{u}^{-1}(\{-\mathfrak{r}\} \times V_-)$. Let \hat{A} be the orient blow up of A around all the punctures of A , i.e. $\hat{A} = A \sqcup B_2$ with $B_2 := \sqcup S^1$ being the disjoint union of circles introduced by orient blow up. Hence we have $\partial \hat{A} = B_1 \sqcup B_2$. We choose the orientation of B_1 to be the boundary orientation from \hat{A} , while we choose the orientation of B_2 to be the reverse orientation of the boundary orientation from \hat{A} .

From now on, let us restrict ourselves to $(-\infty, -R] \times V_-$ and make R large when necessary. We say that a 2-form Δ is J -positive, if for a sufficiently large R , Δ is positive on any J -complex planes of $T((-\infty, -R] \times V_-)$ including infinity. In other words, it means that $\inf \Delta(\zeta, J\zeta) > 0$, where the infimum is taken over all the vector ζ 's on $(-\infty, -R] \times V_-$ with norm $\|\zeta\|_{g_{W_-}} = 1$ (Recall that g_{W_-} is a translational invariant metric).

From (5) and (6) we know that $Pd\lambda_{-\infty} + Qdr \wedge \lambda_{-\infty}$ is $J_{-\infty}$ -positive for any constants $P, Q > 0$. In particular, $d\lambda_{-\infty} + dr \wedge \lambda_{-\infty}$ is $J_{-\infty}$ -positive. Therefore, we could get that $d\lambda_{-\infty} + dr \wedge \lambda_{-\infty}$ is J -positive. Since $d\lambda_{-\infty}$ is $J_{-\infty}$ -non-negative, there exist $C_1, \kappa_1 > 0$ such that

$$d\lambda_{-\infty} + C_1 e^{\kappa_1 r} (dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})$$

is J -positive, so is

$$d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty},$$

where the constant $C_1 > 0$ depends on R , and we can choose C_1 close to 0 by making R large,

Therefore, restricted to J -complex planes in $T((-\infty, -R] \times V_-)$ for large R , we have

$$\omega \leq C_2 \left(d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right), \quad (9)$$

where the constant $C_2 \geq 1$ depends on R , and we can choose C_2 close to 1 by making R large.

Similarly, since $dr \wedge \lambda_{-\infty}$ is $J_{-\infty}$ -non-negative, we can see that

$$dr \wedge \lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} d\lambda_{-\infty}$$

is J -positive.

Therefore, when restricted to J -complex planes in $T((-\infty, -R] \times V_-)$ for large R , we have

$$\sigma \wedge \lambda \leq C_2 \left(dr \wedge \lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} d\lambda_{-\infty} \right). \quad (10)$$

On the other hand, since $P\omega + Q\sigma \wedge \lambda$ is J -positive for any $P, Q > 0$, when restricted on J -complex planes in $T((-\infty, -R] \times V_-)$ for large R , we obtain

$$|dr \wedge \lambda_{-\infty}| \leq C_2 (e^{\kappa_1 r} \omega + \sigma \wedge \lambda), \quad (11)$$

and

$$|d\lambda_{-\infty}| \leq C_2 (\omega + e^{\kappa_1 r} \sigma \wedge \lambda). \quad (12)$$

Therefore, we have

$$\begin{aligned} & \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \\ \leq & \int_{\hat{A}} \tilde{u}^* \omega + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \\ \leq & C_2 \int_{\hat{A}} \tilde{u}^* \left(d\lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \\ \leq & C_2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} - C_2 \int_{B_2} \tilde{u}^* \lambda_{-\infty} \\ & + C_2 \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \\ \leq & C_2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} + C_2 \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \\ & + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega. \end{aligned} \quad (13)$$

While,

$$\begin{aligned} & \left| \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \\ \leq & \int_{\hat{A}} \left| \tilde{u}^* \left(\frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \\ \leq & C_1 C_2 \int_{\hat{A}} |\tilde{u}^* e^{\kappa_1 r} (e^{\kappa_1 r} \omega + \sigma \wedge \lambda)| \\ \leq & \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + C_1 C_2 \kappa_1^{-1} e^{-\kappa_1 \mathfrak{r}} \int_{\hat{A}} \tilde{u}^* (\kappa_1 e^{\kappa_1(\mathfrak{r}+r)} \sigma \wedge \lambda). \end{aligned} \quad (14)$$

Since $\int_{-\infty}^{-\mathfrak{r}} \kappa_1 e^{\kappa_1(\mathfrak{r}+r)} dr = 1$, we have

$$\int_{\hat{A}} \tilde{u}^* (\kappa_1 e^{\kappa_1(\mathfrak{r}+r)} \sigma \wedge \lambda) \leq E_\lambda(\tilde{u}).$$

Therefore, by picking R sufficiently large, we can make \mathfrak{r} sufficiently large, and then (14) implies

$$\left| \int_{\hat{A}} \tilde{u}^* \left(\frac{C_1 e^{\delta_1 r}}{1 + C_1 e^{\delta_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \leq \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{1}{4} E_\lambda(\tilde{u}|_{W_-}). \quad (15)$$

Let $\Phi(r) = \int_{-\infty}^r \phi(t)dt$, and then we get

$$\begin{aligned}
& \int_{\tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) \\
\leq & \int_{\hat{A}} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) \\
\leq & C_2 \int_{\hat{A}} \tilde{u}^* \left(\phi(r)dr \wedge \lambda_{-\infty} + \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r)d\lambda_{-\infty} \right) \\
& + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) \\
\leq & C_2 \int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) - C_2 \int_{\hat{A}} \tilde{u}^* (\Phi(r)d\lambda_{-\infty}) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r)d\lambda_{-\infty} \right) + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) \\
\leq & C_2 \int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) - C_2 \left\{ \int_{\hat{A}} \tilde{u}^* \left[\Phi(r)d\lambda_{-\infty} + \Phi(r) \frac{C_1 e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right] \right\} \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r)d\lambda_{-\infty} \right) \\
& + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) \\
\leq & C_2 \int_{\hat{A}} \tilde{u}^* d(\Phi(r)\lambda_{-\infty}) + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r)d\lambda_{-\infty} \right) + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda) \\
= & C_2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \\
& + C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r)d\lambda_{-\infty} \right) \\
& + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r)\sigma \wedge \lambda). \tag{16}
\end{aligned}$$

While we have

$$\begin{aligned}
& \left| C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\Phi(r) \frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} dr \wedge \lambda_{-\infty} \right) \right| \\
\leq & C_2 C_1 \int_{\hat{A}} \tilde{u}^* |e^{\kappa_1 r} dr \wedge \lambda_{-\infty}| \\
\leq & \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{1}{4} E_\lambda(\tilde{u}|_{W_-}), \tag{17}
\end{aligned}$$

and

$$\begin{aligned}
& \left| C_2 C_1 \int_{\hat{A}} \tilde{u}^* \left(\frac{e^{\kappa_1 r}}{1 + C_1 e^{\kappa_1 r}} \phi(r) d\lambda_{-\infty} \right) \right| \\
& \leq C_2^2 C_1 \int_{\hat{A}} \tilde{u}^* e^{\kappa_1 r} (\omega + e^{\kappa_1 r} \sigma \wedge \lambda) \\
& \leq \frac{1}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{1}{4} E_\lambda(\tilde{u}|_{W_-}).
\end{aligned} \tag{18}$$

Therefore, from (13), (15), (16), (17), and (18), we get

$$\begin{aligned}
E(\tilde{u}|_{W_-}) &:= E_\omega(\tilde{u}|_{W_-}) + E_\lambda(\tilde{u}|_{W_-}) \\
&\leq 2C_2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} + \frac{3}{4} E_\omega(\tilde{u}|_{W_-}) + \frac{3}{4} E_\lambda(\tilde{u}|_{W_-}) \\
&\quad + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega + \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r) \sigma \wedge \lambda).
\end{aligned}$$

Thus,

$$\begin{aligned}
E(\tilde{u}|_{W_-}) &\leq 8C_2 \int_{B_1} \tilde{u}^* \lambda_{-\infty} + 4 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \\
&\quad + 4 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\phi(r) \sigma \wedge \lambda).
\end{aligned}$$

Now we define a function τ by $\tau(r) = \frac{R+r}{R-\mathbf{r}}$ for $-\mathbf{r} \leq r \leq -R$. Since $\tau(-\mathbf{r}) = 1$ and $\tau(-R) = 0$, by Stokes' Theorem we get

$$\begin{aligned}
\left| \int_{B_1} \tilde{u}^* \lambda_{-\infty} \right| &= \left| \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* d(\tau(r) \lambda_{-\infty}) \right| \\
&\leq \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} |\tilde{u}^* d(\tau(r) \lambda_{-\infty})| \\
&\leq C_3 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega'
\end{aligned}$$

where C_3 is a constant depends on R , and the last inequality follows from the fact that on any J -complex planes the symplectic form ω' is positive. For the same reason, we also have

$$\int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega \leq C_3 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega'$$

and

$$\int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* (\sigma \wedge \lambda) \leq C_3 \int_{\{\Sigma \setminus A\} \cap \tilde{u}^{-1}(W_-)} \tilde{u}^* \omega'.$$

These imply

$$E(\tilde{u}|_{W_-}) \leq C_4 E_{\text{symp}, 2R}(\tilde{u}), \quad (19)$$

where C_4 is a constant independent of \tilde{u} .

For positive end W_+ , it is similar. Indeed, based on the proof for the negative end, one can show that there exists a constant $C_5 > 0$ independent of \tilde{u} , such that

$$E(\tilde{u}|_{W_+}) = E_\omega(\tilde{u}|_{W_+}) + E_\lambda(\tilde{u}|_{W_+}) \leq C_5 \left(\sum \int \gamma_+^* \lambda_{+\infty} + E_{\text{symp}, 2R}(\tilde{u}) \right), \quad (20)$$

where the summation is taken over all the periodic orbits γ_+ 's of $\mathbf{R}_{+\infty}$ to which \tilde{u} converges at positive infinity.

By (20) and (19), we have $E_a(\tilde{u}) \leq C \left(\sum \int \gamma_+^* \lambda_{+\infty} + E_{\text{symp}, a}(\tilde{u}) \right)$. \square

4 An application to closed symplectic manifolds with a compatible J

Now we would like to apply the previous results to study the moduli space of J -holomorphic curves passing through a fix point in a closed symplectic manifold. This generalizes some results in [3].

Let M be a closed smooth symplectic manifold of dimension $2N$ with symplectic form ω' , and J be a compatible almost complex structure. For a sufficiently small neighborhood U of $p \in M$, there exists a Darboux coordinate chart $\varphi : U \rightarrow B(O, \epsilon) \subseteq \mathbb{C}^N$ such that $\varphi(p) = O$, $\varphi_* J|_O = i|_O$ and $\varphi^* \omega_{st} = \omega'$, where O is the origin, $B(O, \epsilon) := \{z \in \mathbb{C}^N \mid |z| < \epsilon\}$ and i is the standard complex structure on \mathbb{C}^N , and $\omega_{st} := \frac{i}{2} \sum_{k=1}^n dz_k \wedge d\bar{z}_k = \sum_{k=1}^n dx_k \wedge dy_k$ is the standard symplectic structure on \mathbb{C}^N . We identify $B(O, \epsilon) \setminus O$ with $W_- := \mathbb{R}^- \times S^{2N-1}$ via the map $\psi(z) = (\log |z| - \log \epsilon, \frac{z}{|z|})$. Let us simplify the notation $(\psi \circ \varphi)_* J$ by J when there is no confusion.

We define ξ , \mathbf{R} , λ , and σ as before. Then $\lambda_{-\infty} := \lim_{s \rightarrow -\infty} f_s^* \lambda = \Pi^* \lambda_{st}$, where

$$\lambda_{st} = \frac{1}{2} \sum_{k=1}^N (x_k dy_k - y_k dx_k) \Big|_{S^{2N-1}}$$

is the standard contact 1-form on the unit sphere $S^{2N-1} \subseteq \mathbb{C}^N$, and $\Pi : \mathbb{R}^- \times S^{2N-1} \rightarrow S^{2N-1}$ is the projection. Define the 2-form ω on W_- by $\omega(u, v) = (2dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})(\pi_\xi u, \pi_\xi v)$, where $u, v \in T_{(r, \theta)} W_-$, and π_ξ is the projection: $TW_- = \mathbb{R}(\frac{\partial}{\partial r}) \oplus \mathbb{R}(\mathbf{R}) \oplus \xi \rightarrow \xi$. Using the facts that $\omega_{-\infty} := \lim_{s \rightarrow -\infty} f_s^* \omega = d\lambda_{-\infty}$ and $(\psi \circ \varphi)_* \omega' = \omega_{st} = d(e^{2r} \lambda_{-\infty}) = e^{2r} (2dr \wedge \lambda_{-\infty} + d\lambda_{-\infty})$, we can easily check that (W_-, ω, J) satisfies (AC1)-(AC7). Notice also that $\mathbf{R}_{-\infty} := \lim_{s \rightarrow -\infty} f_s^* \mathbf{R}$ restricted to S^{2N-1} is exactly the standard Reeb vector field on

S^{2N-1} , so we can see that J is Morse-Bott. Therefore, $(M \setminus p, \omega, J)$ an almost complex manifold with an asymptotically cylindrical negative end of contact type.

Let (Σ, j) be a Riemann surface with finitely many punctures and $\tilde{u} : \Sigma \rightarrow M \setminus p$ be a J -holomorphic curve, i.e. $J(\tilde{u}) \circ T\tilde{u} = T\tilde{u} \circ j$.

We say a puncture q of Σ is removable if around q , \tilde{u} converges to a point in $M \setminus p$. Otherwise, we say q is non-removable. To clarify the relations between different concepts we state the following lemma.

Lemma 11. *Suppose that all the punctures of Σ are non-removable. Then the following statements are equivalent.*

1. \tilde{u} converges to some Reeb orbits of $\mathbf{R}_{-\infty}$ at negative infinity around the punctures of Σ .
2. $E_a(\tilde{u})$ is finite for any $a \geq 0$.
3. $E_a(\tilde{u})$ is finite for some $a \geq 0$.
4. $\lim_{a \rightarrow +\infty} E_{\text{symp}, a}(\tilde{u})$ is finite.
5. If we view \tilde{u} as a map from Σ to M , then \tilde{u} extends smoothly over S , where S is the smooth Riemann surface associated to Σ .

Proof. It is obvious that (2) \iff (3). Lemma 7 says (1) \implies (3). From Theorem 6 and Removable Singularity Theorem, we get (3) \implies (1). (1) \implies (4) follows from direct calculation. (4) \implies (5) is true by the Removable Singularity Theorem. Finally, (5) \implies (1) is guaranteed by Theorem B² in [12]. \square

Assuming any of the (1)-(5) is true, then by (4) and (5) we have

$$E_{\text{symp}}(\tilde{u}) = \lim_{a \rightarrow +\infty} E_{\text{symp}, a}(\tilde{u}) = \lim_{a \rightarrow +\infty} \int_{\tilde{u}^{-1}(E \setminus W_-^a)} \tilde{u}^* \omega' = \int_S \tilde{u}^* \omega' < +\infty.$$

Thus, $E(\tilde{u}) = E_{\text{symp}}(\tilde{u}) + E_\omega(\tilde{u}) + E_\lambda(\tilde{u})$ is well defined.

The multiplicity of a Reeb orbit γ is the degree of γ as a cover of a simple Reeb orbit. For each non-removable punctures q of Σ , we can associate a positive integer which is the multiplicity of the corresponding Reeb orbit that \tilde{u} converges to around q . The next proposition says that the number of non-removable punctures of \tilde{u} counted with multiplicity is bounded by a constant independent of \tilde{u} and (Σ, j) .

Proposition 12. *Given $Q > 0$, there exists a number $N \in \mathbb{N}$ such that for any finitely punctured Riemann surface (Σ, j) , and any non-constant J -holomorphic map $\tilde{u} : \Sigma \rightarrow M \setminus p$ with $E(\tilde{u}) \leq Q$, the number of non-removable punctures of \tilde{u} counted with multiplicity is no greater than N .*

²Theorem B is stated for the case of a J -holomorphic strip with Lagrangian boundary condition, but it is easy to see that it is also true in this closed case.

Proof. Suppose to the contrary. Let \tilde{u}_n be a J -holomorphic curve from a finitely punctured Riemann surface Σ_n to $M \setminus p$ with $E(\tilde{u}_n) \leq E$, such that the number of non-removable punctures of \tilde{u}_n goes to infinity as $n \rightarrow \infty$.

Let $\{q_n^m\}_{1 \leq m \leq m_n}$ be the set of non-removable punctures of Σ_n , and $\{\gamma^{q_n^m}\}_{1 \leq m \leq m_n}$ be the Reeb orbits of $\mathbf{R}_{-\infty}$ to which \tilde{u}_n converges around the non-removable puncture q_n^m . We denote the period of $\gamma^{q_n^m}$ by $2k_{q_n^m}\pi$, with $k_{q_n^m} \in \mathbb{Z}_{>0}$. By assumption, we get that

$$\sum_{m=1}^{m_n} k_{q_n^m} \rightarrow \infty$$

as $n \rightarrow \infty$.

We can pick $[r_n - 1, r_n + 1] \times S^{2N-1}$ inside W_- such that

$$\tilde{u}_n^{-1} \{[r_n - 1, r_n + 1] \times S^{2N-1}\}$$

consists of connected components $A_n^m \subset \Sigma_n$ for $1 \leq m \leq m_n$ with

$$\partial A_n^m = \partial_1 A_n^m - \partial_2 A_n^m,$$

$$\tilde{u}_n(\partial_1 A_n^m) \subset \{r_n + 1\} \times S^{2N-1},$$

$$\tilde{u}_n(\partial_2 A_n^m) \subset \{r_n - 1\} \times S^{2N-1}.$$

Pick $\phi_n(r) \in \mathcal{C}_-$ to be a function satisfying $\phi(r) = \frac{1}{3}$ for $r_n - 1 \leq r \leq r_n + 1$. Define a 2-form Ω_n by $\Omega_n := \phi_n(r)\sigma \wedge \lambda + \omega$ on W_- . We know Ω_n is non-degenerate over $[r_n - 1, r_n + 1] \times S^{2N-1}$. Since S^{2N-1} is compact, by choosing r_n sufficiently negative, the Gromov's Monotonicity Theorem implies $\int_{A_n^m} \tilde{u}_n^* \Omega_n > k_{q_n^m} \delta_0 > 0$, for some δ_0 independent of m and n . Therefore, we get

$$\begin{aligned} E &\geq E(\tilde{u}_n) \\ &\geq E_\omega(\tilde{u}_n) + E_\lambda(\tilde{u}_n) \\ &= \int_{\tilde{u}_n^{-1}(W_-)} \tilde{u}_n^* \omega + \sup_{\phi \in \mathcal{C}_-} \int_{\tilde{u}_n^{-1}(W_-)} \tilde{u}_n^*(\phi(r)\sigma \wedge \lambda) \\ &\geq \int_{\tilde{u}_n^{-1}(W_-)} \tilde{u}_n^* \omega + \int_{\tilde{u}_n^{-1}(W_-)} \tilde{u}_n^*(\phi_n(r)\sigma \wedge \lambda) \\ &\geq \int_{\tilde{u}_n^{-1}([r_n-1, r_n+1] \times S^{2N-1})} \tilde{u}_n^* \Omega_n \\ &= \sum_{m=1}^{m_n} \int_{A_n^m} \tilde{u}_n^* \Omega_n \\ &\geq \sum_{m=1}^{m_n} k_{q_n^m} \delta_0 \\ &\rightarrow \infty. \end{aligned}$$

□

Let \tilde{u} be a non-constant J -holomorphic curve from a smooth Riemann surface (S, j) to M . By the Carleman Similarity principle, we know $\tilde{u}^{-1}(p)$ is discrete, and hence finite. Let (Σ, j) be the punctured Riemann surface $(S \setminus \tilde{u}^{-1}(p), j)$. Now \tilde{u} can be viewed as a J -holomorphic curve from Σ to $M \setminus p$. This means that the condition (4) in Lemma 11 is satisfied, so we have (1)-(5). Now it follows immediately from Theorem 10 that

Proposition 13. *There exists $C > 0$, such that for any Riemann surface (S, j) and any non-constant J -holomorphic curve $\tilde{u} : S \rightarrow M$, we have $E(\tilde{u}) \leq CE_{\text{symp}}(\tilde{u})$.*

Proposition 12 and Proposition 13 imply the following

Corollary 14. *Given $Q > 0$, there exists $N \in \mathbb{N}$, such that for any smooth Riemann surface (S, j) and any non-constant J -holomorphic curve $\tilde{u} : S \rightarrow M$ with symplectic area $E_{\text{symp}}(\tilde{u}) \leq Q$, we have that the number of pre-images of p counted with multiplicity is less than N .*

Remark 15. Note that N is independent of the genus and the complex structure of the Riemann surface (S, j) , and the map \tilde{u} itself.

Let $\mathcal{M}_g(M, J, Q)$ be the moduli space of stable J -holomorphic curves \tilde{u} in M with genus g and $E_{\text{symp}}(\tilde{u}) \leq Q$. From Corollary 14 and Theorem 9, we can compactify $\mathcal{M}_g(M, J, Q)$ by including holomorphic buildings (See [3] for more discussions).

It will be very interesting and useful to generalize the results in this paper by replacing the fixed point p with an almost complex submanifold.

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